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Period-Doubling Bifurcations and Chaotic Motion for a Parametrically Forced Pendulum

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A parametrically forced pendulum is studied numerically both with and without friction. In both cases, period-doubling sequences of bifurcation are found. In the dissipative case, the period-doubling sequence leads to strange attractors, while in the conservative case, the sequence is responsible for the destruction of stable zones.

KEY WORDS: Dissipative; measure-preserving; period-doubling bifurcation; stochastic; strange attractor.

1. INTRODUCTION

Many papers dealing with strange attractors have appeared in the literature recently. However, many of the systems which have been studied have a somewhat tenuous connection with physical reality. In this paper, the onset of chaotic motion in a very simple, but real, physical system will be discussed. Specifically, a parametrically forced pendulum will be treated both with and without air friction. The same sequence of bifurcations can be seen in both cases as the external forcing is increased.

In the system to be discussed, the appearance of strange attractors is preceded by an infinite sequence of period-doubling bifurcations. This is interesting in light of Feigenbaum's[1] discovery that such sequences have a universal character, at least for one-dimensional maps, as well as more recent work which suggests that the universality may extend to more complex systems.[2–4] Another interesting result to be reported is the fact

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that this same sequence occurs even in the limit of no dissipation. At the end of the sequence, the frictionless pendulum wanders over a large region of the Poincaré map.

When dissipation is present, the sequence of bifurcations leads to the appearance of strange attractors. These attractors go through a process of coalescence of the type discussed by Simo[5] for Hénon's map.[6] For forces slightly beyond the period-doubling range, one finds strange attractors in which the momentum of the pendulum does not change sign. After the process of coalescence, one has two distinct strange attractors, each of which consists of two pieces. Furthermore, each of the attractors is still associated with a definite sign of the momentum. Finally, when the force is increased further, the sign of the momentum begins to switch. This sign reversal occurs infrequently for forces which are slightly above the threshold, but becomes increasingly frequent as the force is increased.

If one lets the dissipation become small, the thresholds for the period-doubling bifurcations decrease, but they converge to finite values. Thus, the period-doubling sequence also occurs in the conservative case.

In the conservative case, one has a stochastic zone near the separatrix of the unperturbed pendulum. This zone exists even for small forcing levels and is produced by homoclinic intersections between the stable and unstable manifolds of the point of inverted equilibrium. For low forcing levels, one has stable periodic orbits surrounded by KAM surfaces inside the "separatrix." As the period-doubling sequence occurs, these stable regions become smaller and stochastic motion occupies a greater fraction of the Poincaré map.

2. THE SYSTEM

Let us consider a parametrically forced pendulum described by

$$\frac{d^2\phi}{dt^2} + k\frac{d\phi}{dt} + \left[a + 2q\cos(\Omega t)\right]\sin\phi = 0 \tag{1}$$

In Eq. (1), ϕ is the angle measured from the direction of gravity, k represents the effect of air resistance, a is the square of the pendulum's natural frequency, q is proportional to the amplitude of the vertical motion of the point of support,[7] and Ω is the frequency of the vertical vibration. It is well known from the theory of Mathieu equations that the equilibrium point $\phi = 0$ becomes unstable for certain values of the parameters in Eq. (1). For example, if one takes $k = 0$ and $\Omega = 2$, there are resonance zones in the a–q plane starting at the points $q = 0$, $a = n^2$. Thus, for $a = 1$, $\Omega = 2$, and $k = 0$, the equilibrium point $\phi = 0$ is unstable for arbitrarily small values of q. In the calculations to be reported in the following section, $a = 1$ and $\Omega = 2$.

Let us introduce the momentum of the pendulum so that Eq. (1) can be written as two first-order ODEs:

$$d\phi/dt = p \tag{2}$$

$$dp/dt = -kp - \left[1 + 2q\cos(2t)\right]\sin\phi \tag{3}$$

In Eqs. (2) and (3), the restriction to $a = 1$ and $\Omega = 2$ has been made. The Poincaré map of the system is determined by plotting the values of ϕ and p for $t = n\pi$, where n is a nonnegative integer. The equilibrium points of the pendulum show up as fixed points in the Poincaré map at $\phi = 0$, $p = 0$ and $\phi = \pm\pi$, $p = 0$. The equilibrium point at $\phi = \pm\pi$, $p = 0$ is unstable at all the values of k and q for which computations are reported in the following sections. Indeed, the homoclinic intersections between the stable and unstable manifolds of this point play an important role in the chaotic motion observed in both the conservative and dissipative cases.

3. RESULTS

When k is positive, q must exceed a certain threshold in order to destabilize the equilibrium point $\phi = 0$. When k is small compared to unity, it is easily shown[7] that the threshold for parametric resonance is

$$q = k \tag{4}$$

When q exceeds this threshold, finite-amplitude motion results.

Let us first consider the case $k = 0.2$. A vibration having period 2π bifurcates from the origin when q exceeds 0.2 and this vibration is the only stable solution up to $q = 0.713 \pm 0.012$. At this value of q, two distinct rotational motions having period π appear (one clockwise and the other counterclockwise). Each of these periodic solutions goes through a period-doubling sequence of bifurcations as q is increased. The thresholds for solutions of period 2π, 4π, 8π, 16π, and 32π are $q = 0.7925 \pm 0.0025$, 0.984 ± 0.010, 1.01975 ± 0.00025, 1.03075 ± 0.00025, and 1.03275 ± 0.00025. Thus,

$$\begin{aligned} \frac{q_4 - q_2}{q_8 - q_4} &= 5.35 \pm 0.30 \\ \frac{q_8 - q_4}{q_{16} - q_8} &= 3.25 \pm 0.25 \\ \frac{q_{16} - q_8}{q_{32} - q_{16}} &= 5.6 \pm 1.0 \end{aligned} \tag{5}$$

It therefore appears possible that this sequence is described by the Feigenbaum constant (4.669 . . .).

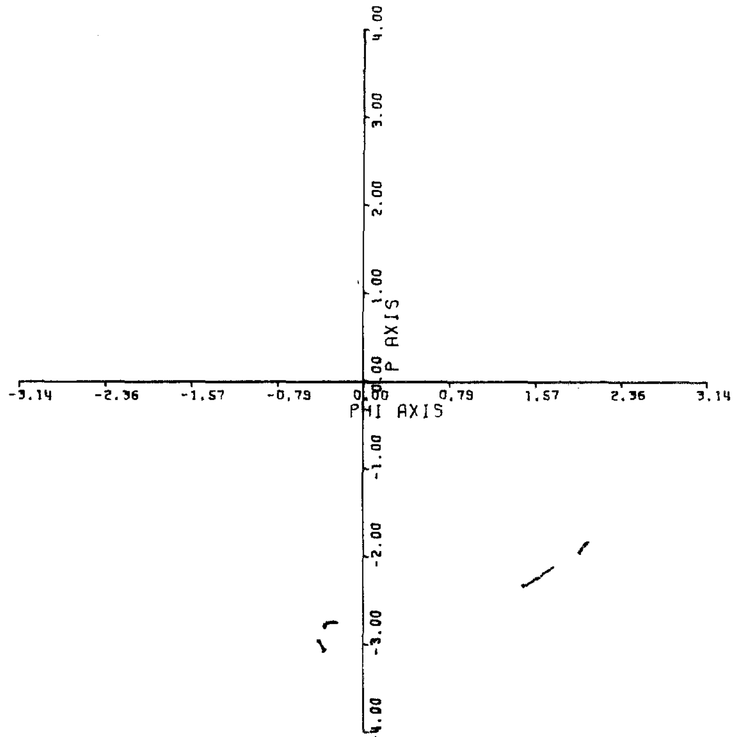

Fig. 1. $q = 1.036$, $k = 0.2$, $\phi(0) = 1.036$, $p(0) = -2.150$, 200 points.

The end result of the cascade of bifurcations is a pair of strange attractors. These attractors also go through a remarkable series of transitions as q is increased. At the lowest value of q for which strange attractors exist, the Poincaré maps of the attractors seem to consist of Cantor sets of points. As q is increased, the different parts of the attractors coalesce and form larger pieces. For example, in Fig. 1, the strange attractor is seen to be composed of four distinct pieces for $q = 1.036$. In Fig. 2, these pieces have coalesced to form two large pieces for $q = 1.0375$. Simo[5] has observed this phenomenon in Hénon's map[6] and explained the coalescence in terms of heteroclinic intersections between the stable and unstable manifolds of the periodic points which are created by the cascade of period-doubling bifurcations.

It should be noted that the attractors in Figs. 1 and 2 correspond to clockwise rotations (the sign of the momentum is negative). Thus, the number of rotations executed by the pendulum increases roughly linearly as a function of time. When q exceeds 1.045, the momentum of the pendulum

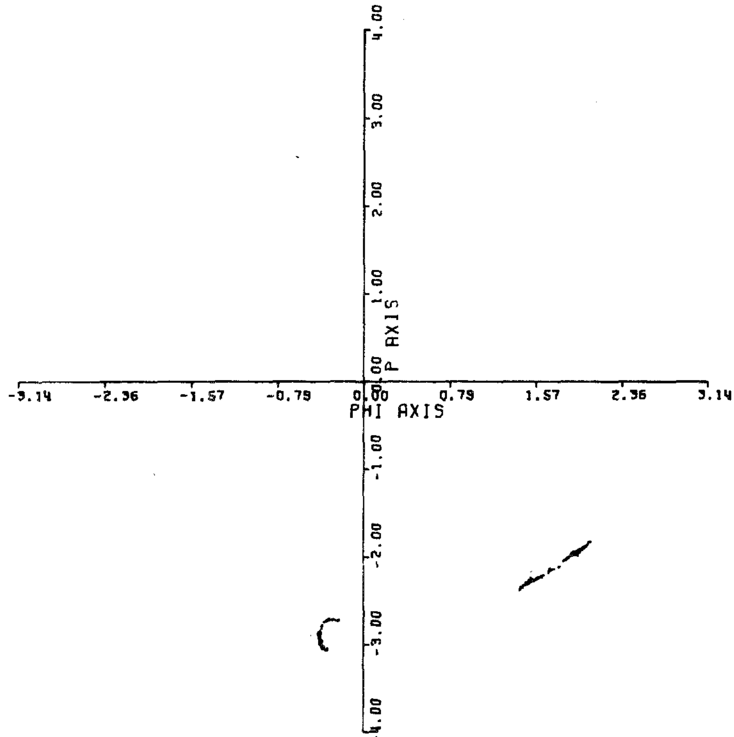

Fig. 2. $q = 1.0375$, $k = 0.2$, $\phi(0) = 2.063$, $p(0) = -1.819$, 200 points.

occasionally switches sign. When q is only slightly above this threshold, these reversals are infrequent. In fact, as q approaches 1.045 from above, the frequency of reversals appears to approach infinity. For example, for $q = 1.0455$, a run was made for a total time of 500π, and the momentum was positive between 91π and 416π. For $q = 1.046$, the longest interval between sign reversals was 193π. For $q = 1.05$, the longest interval was 123π, and for $q = 1.1$, the longest interval was 56π. The effect of these sign reversals is to cause the net number of rotations to increase more slowly as a function of time as q is increased.

In Figs. 3 and 4, the Poincaré map of the strange attractor is shown for $q = 1.05$ and 1.1. The strange attractor appears to be closely related to the unstable manifold of the fixed point $\phi = \pi$, $p = 0$. A similar phenomenon has been observed by Holmes in a model of a periodically forced beam.[8] A reasonable explanation for the onset of rotation reversals would be that there is a heteroclinic intersection between the stable and unstable manifolds of some of the periodic points corresponding to clockwise and

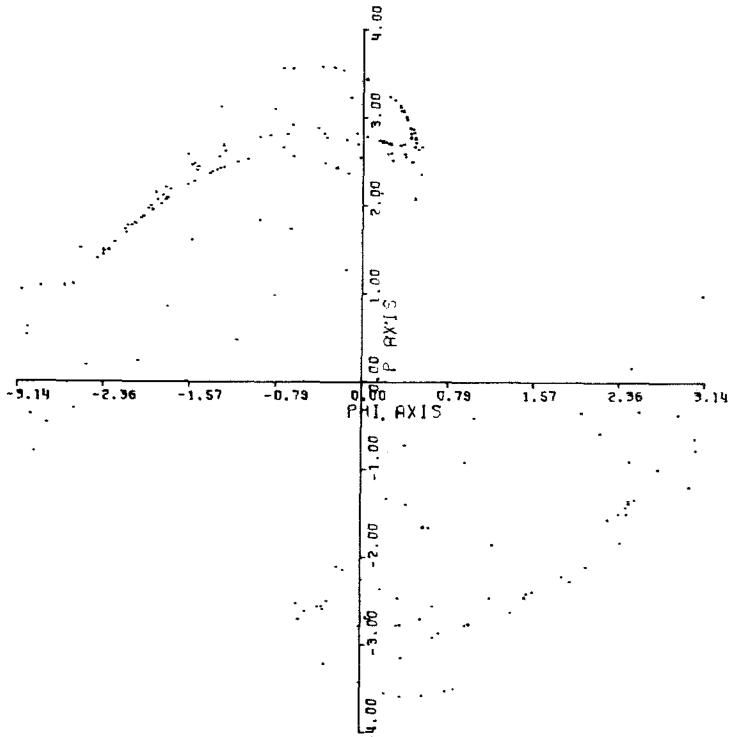

Fig. 3. $q = 1.05$, $k = 0.2$, $\phi(0) = 0.3456$, $p(0) = 3.104$, 200 points.

counterclockwise rotations. This would form a bridge between the two distinct attractors which exist at lower values of q and allow reversals. Unfortunately, there are an infinite number of periodic points in the Poincaré map, and looking for such heteroclinic intersections would be difficult unless one had an intuitive guide as to which points are important.

The strange attractor persists up to $q = 1.58$. From this point on, there are two stable solutions—clockwise and counterclockwise rotations having period π. Thus, the system reverts to periodicity at large q. Similar behavior has been found in the Lorenz model.[9–11]

Let us consider what happens to the above sequence of phenomena as k is decreased. For example, if $k = 0.05$, period-2π vibrations exist up to $q = 0.64$. Period-π rotations first appear at 0.525 ± 0.025. In the interval between 0.525 and 0.64, there are four distinct basins of attraction. Period-2π rotations bifurcate from the period-π rotations at $q = 0.663 \pm 0.012$. The bifurcation points for period-4π and -8π rotations are $0.8405 \pm$

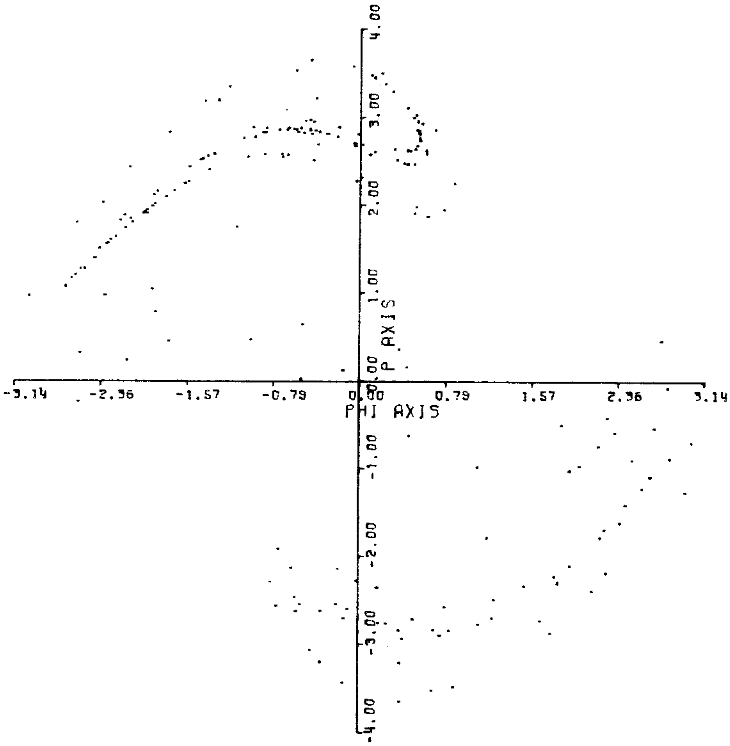

Fig. 4. $q = 1.1$, $k = 0.2$, $\phi(0) = -1.965$, $p(0) = 1.928$, 200 points.

0.0005 and 0.8713 ± 0.0012. Strange attractors with sign reversals exist for $q = 0.90$ (and beyond).

Two observations can be made at this point. First, even though the friction has been reduced by a factor of 4, the bifurcation points in the period-doubling sequence have been lowered by only 15%. This suggests (and we will show below) that, as the friction goes to zero, the bifurcation points converge to finite values (i.e., that the period-doubling bifurcation sequence occurs for the measure-preserving case as well as the dissipative case).

The second observation is that, for $k = 0.05$,

$$\frac{q_4 - q_2}{q_8 - q_4} = 5.8 \pm 0.8 \tag{6}$$

Comparing this value with the corresponding value in Eq. (5), it is seen that the ratio is fairly insensitive to the dissipation.

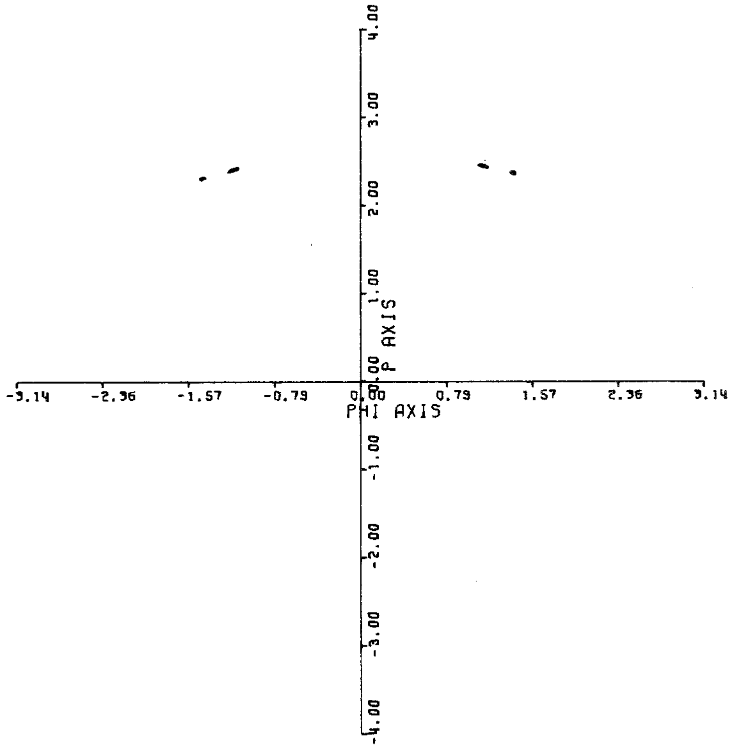

Fig. 5. $q = 0.85$, $k = 0.005$, $\phi(0) = 1.086$, $p(0) = 2.461$, 200 points.

In order to investigate the limit as k goes to zero, q was chosen to be 0.85. When $k = 0.05$, a pair of stable period-4π rotations exist for this value of q. Using one of the points in the Poincaré map as initial condition, the value of k was changed to 0.01. Another 4π rotation resulted. The largest percentage change in any of the eight coordinates of the points in the Poincaré map was 24%. One of the points in the Poincaré map for the period-4π rotation was then used as initial condition for a run with $k = 0.005$. Once again, a period-4π rotation was found, and the largest percentage change in any of the eight coordinates was 1.8%. The Poincaré map for $k = 0.005$ is shown in Fig. 5. The initial condition for this run was one of the points of the period-4π solution for $k = 0.01$. Nevertheless, the intersections of the orbit look almost like four points in the plot. This demonstrates how little the period-4π attractors change as k is lowered from 0.01 to 0.005.

Finally, one of the points of the period-4π solution for $k = 0.005$ was used as the initial condition for a run with $k = 0$. In this case, the Poincaré

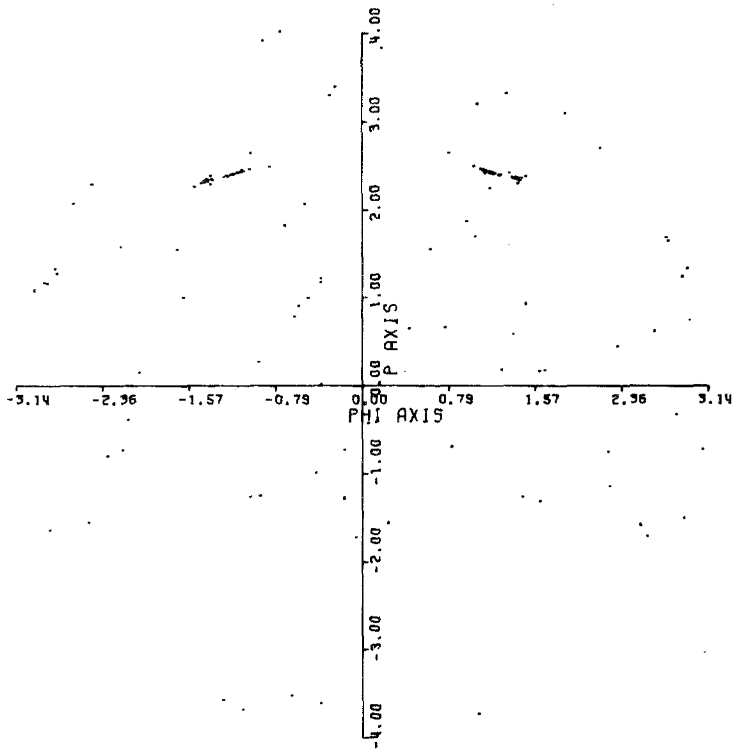

Fig. 6. $q = 0.85$, $k = 0$, $\phi(0) = 1.106$, $p(0) = 2.450$, 200 points.

map is area preserving. The resulting plot is shown in Fig. 6. One can see an island chain consisting of four islands in the plot. The same initial data were used for a run with $q = 0.84$, and the results are plotted in Fig. 7. It is seen that one has an island chain consisting of two islands. This is consistent with the location of the bifurcation points for small but nonzero k.

Let us now consider in more detail the behavior of the system when $k = 0$. For small values of q, one has a stable (elliptic) periodic vibration with period 2π. This orbit is surrounded by KAM surfaces, as one would expect on the basis of KAM theory.[12,13] This stable zone is surrounded by a "fuzzy" separatrix as shown in Fig. 8 for $q = 0.2$. The finite thickness of the separatrix is produced by intersections between the stable and unstable manifolds of the fixed point $\phi = p = 0$.[14] The fixed point $\phi = \pi, p = 0$ also has a fuzzy separatrix as shown in Fig. 9 for $q = 0.2$. In between the inner and outer separatrices, one finds island chains as shown in Fig. 10.

Just as in the dissipative case, a pair of period-π rotational solutions

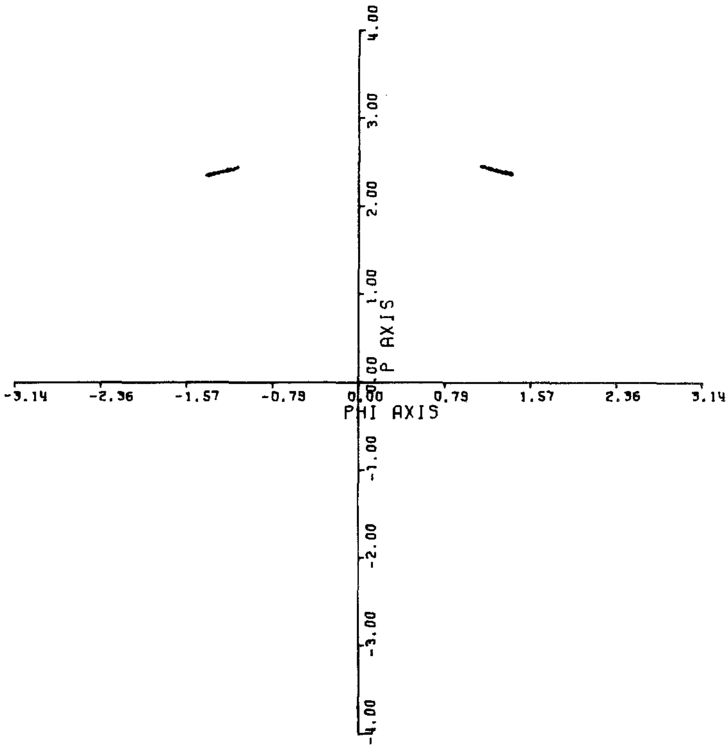

Fig. 7. $q = 0.84$, $k = 0$, $\phi(0) = 1.106$, $p(0) = 2.450$, 200 points.

appears when one increases q. In Fig. 11, a KAM surface surrounding a clockwise rotational solution of period π is shown for $q = 0.6$. As q is increased still further, a 2π rotation bifurcates from each of these solutions. In Fig. 12, one can see an island chain consisting of two islands. The islands contain a counterclockwise period-2π rotation.

It is worth pointing out that the stable area surrounding the periodic solutions decreases considerably as q is increased. This is apparent in Figs. 6 and 12. The stable regions are surrounded by a large stochastic zone. Thus, one starts at around $q = 0.6$ with a pair of fairly large stable zones surrounding period-π solutions. These stable zones disintegrate into smaller and smaller pieces as a result of the period-doubling sequence of bifurcations. Eventually, one is left with a set of saddle points and no stable zones. Of course, other stable zones appear in other parts of the Poincaré maps as q is increased, as is generally the case.[15] Nevertheless, the destruction of a stable zone through period-doubling bifurcations is of interest in itself

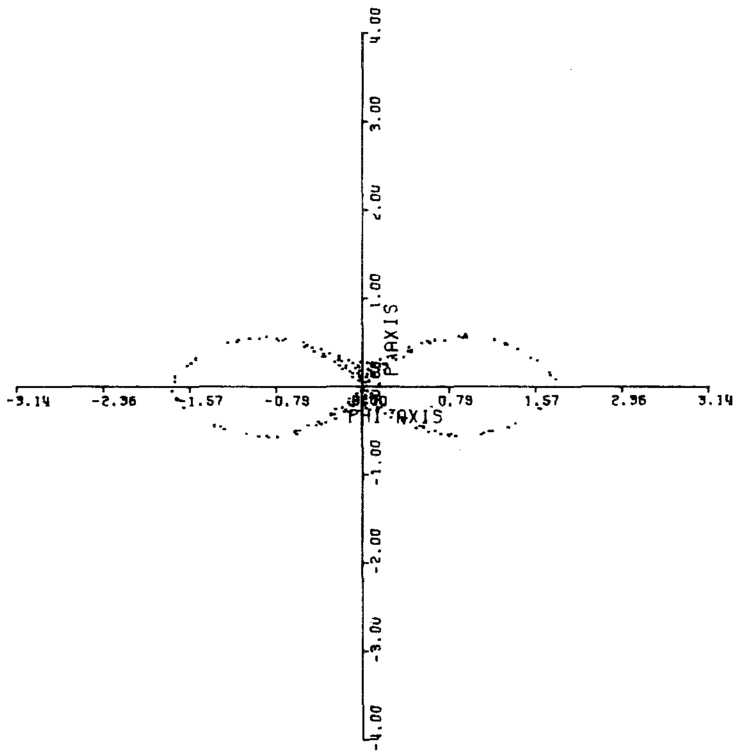

Fig. 8. $q = 0.2$, $k = 0$, $\phi(0) = 0$, $p(0) = -0.1$, 200 points.

because the phenomenon may very well carry over to other conservative systems. If this is true, then in these cases the onset of period doubling can be used as a more quantitative alternative to the "resonance overlap" criterion(14,16) for the destruction of stable zones.

4. CONCLUSION

The results stated above demonstrate the existence of period-doubling sequences of bifurcations in a real mechanical system. The period doubling occurs in both the conservative and dissipative cases, and the periodic solutions which appear as the forcing is varied are continuous functions of the friction as the friction goes to zero.

The period-doubling bifurcations cause the destruction of large stable zones in the conservative case, and the onset of period doubling in

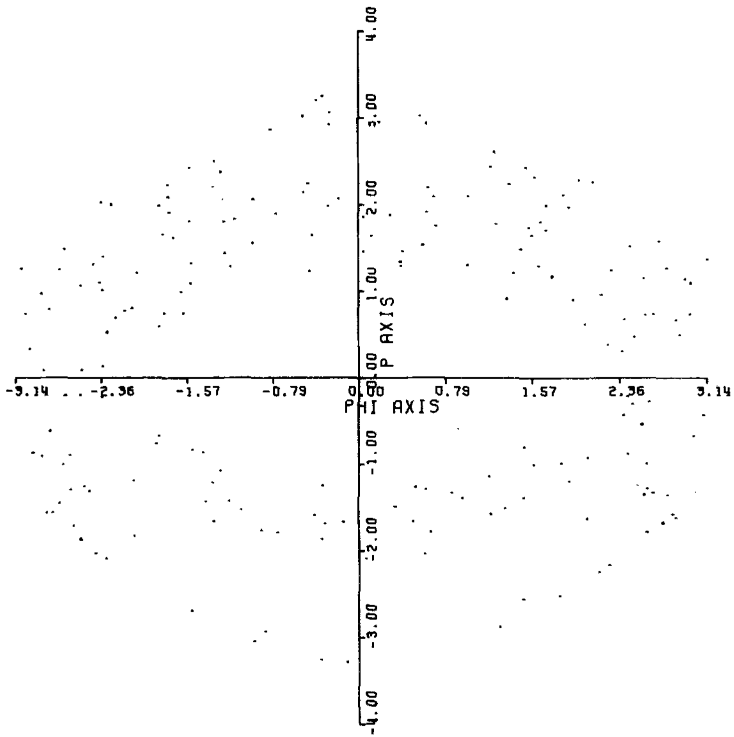

Fig. 9. $q = 0.2$, $k = 0$, $\phi(0) = 0$, $p(0) = -2.0$, 200 points.

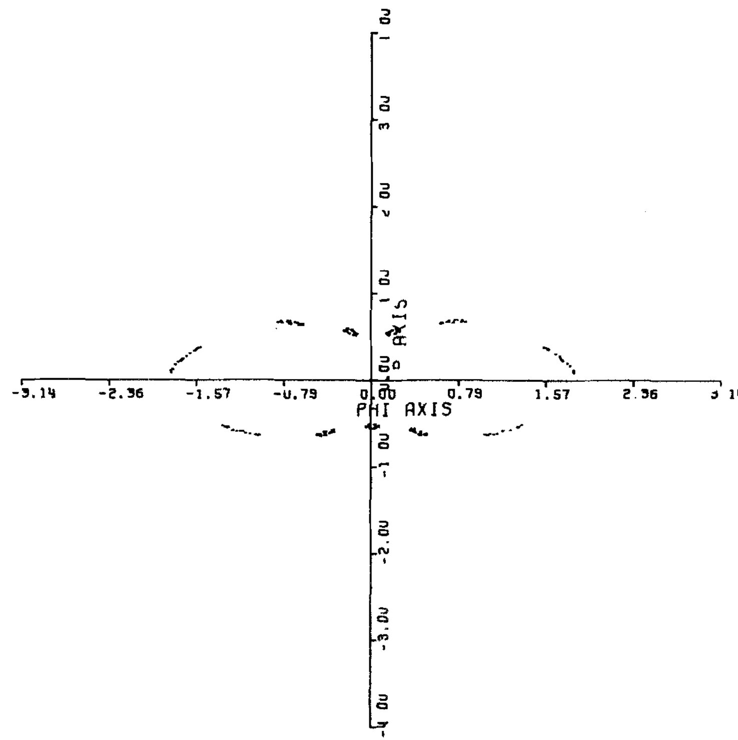

Fig. 10. $q = 0.2$, $k = 0$, $\phi(0) = 0$, $p(0) = -0.5$, 200 points.

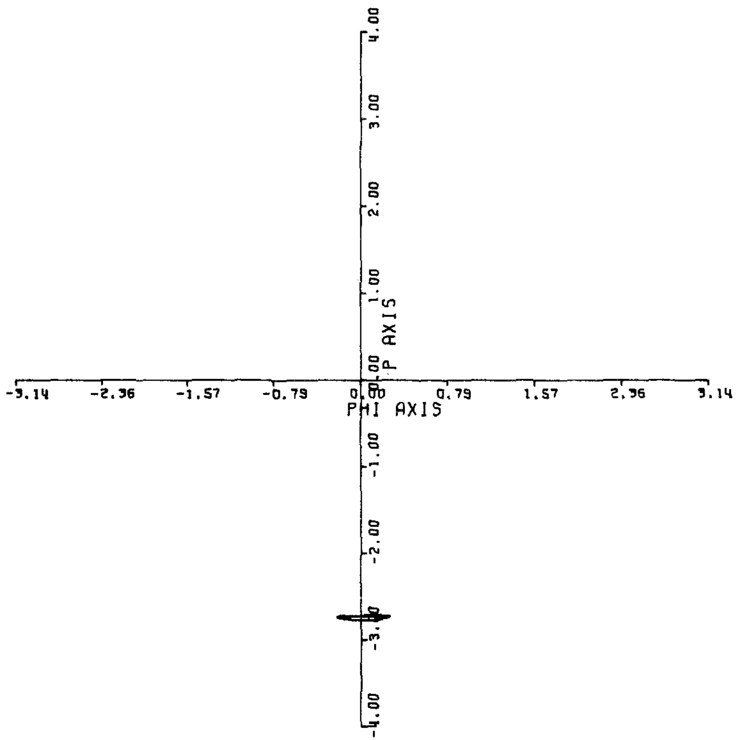

Fig. 11. $q = 0.6$, $k = 0$, $\phi(0) = 0.2397$, $p(0) = -2.730$, 200 points.

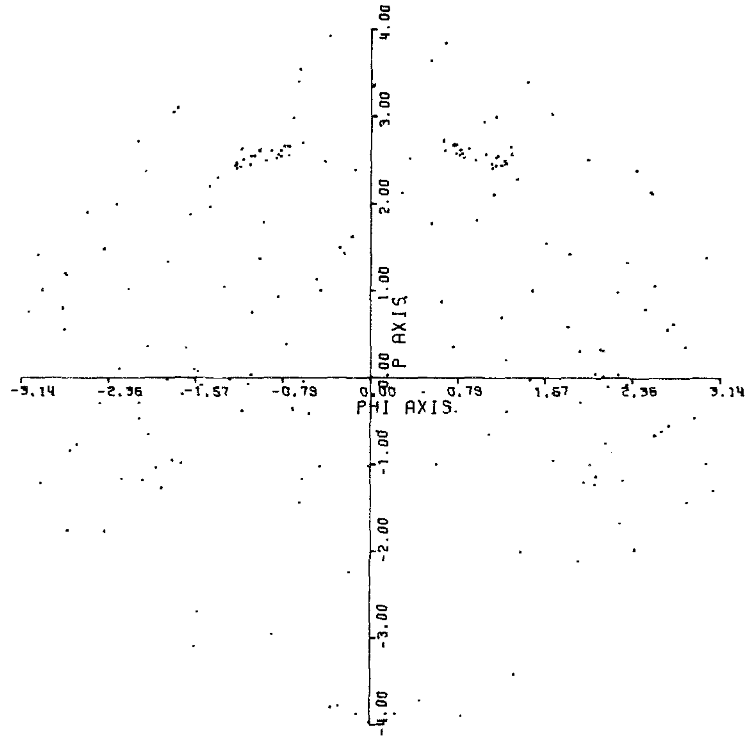

Fig. 12. $q = 0.75$, $k = 0$, $\phi(0) = 0.7196$, $p(0) = 2.667$, 200 points.

conservative systems may well prove to be a replacement for the resonance overlap criterion in many systems.

In the dissipative case, several different kinds of strange attractors appear as the driving force is increased. The first strange attractors to appear have a definite sign of the momentum. They also consist of several disjoint pieces which coalesce as the force is increased in a manner described in Ref. 5 for Hénon's map. Finally, a strange attractor appears in which the momentum reverses sign. This attractor exists over a fairly large range of the driving force, but is eventually replaced by periodic motion at large forces.

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